# Some New Results about The Period of Recurring Decimal

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*Abstract*—**This study mainly discusses period problem of recurring decimals.According to Euler theorem, this paper gives the computation formula of period of recurring decimal, relation of the period and least positive period, and the necessary and sufficient condition that the period is eaqual to least positive period.**

*Keywords-Recurring decimal;Period;Prime.*

## I. INTRODUCTION

We know, irreducible proper fraction can be transformed recurring decimal (pure recurring decimal or mixed recurring decimal), the repetend digit of the recurring decimal is called the period of the recurring decimal, the least repetend digit is called least positive period.

Period problem of the recurring decimal is always a very interesting and difficult problem in number theory; many scholars studied it and derived some beautiful properties of the recurring decimals [1, 2, 5-10]. However, for the period and least positive period of the recurring decimal, there are not computation formulas so far.

Let us now consider 50 fractions between 1/3 and 1/62(besides the fractions which can be denoted as limited decimals), and further convert them into recurring decimals, then their least positive period are as follows.

TABLE 1. FRACTIONS AND LEAST POSITIVE PERIODS

| Fraction | Least positive period | Fraction | Least positive period |
|---|---|---|---|
| 1/3 | 1 | 1/36=1/4*1/9 | 1 |
| 1/6=1/2*1/3 | 1 | 1/37 | 3 |
| 1/7 | 6 | 1/38=1/2*1/19 | 18 |
| 1/9 | 1 | 1/39=1/3*1/13 | 6 |
| 1/11 | 2 | 1/41 | 5 |
| 1/12=1/3*1/4 | 1 | 1/42=1/6*1/7 | 6 |
| 1/13 | 6 | 1/43 | 21 |
| 1/14=1/2*1/7 | 6 | 1/44=1/4*1/11 | 2 |
| 1/15=1/5*1/3 | 1 | 1/45=1/5*1/9 | 1 |
| 1/17 | 16 | 1/46=1/2*1/23 | 22 |
| 1/18=1/2*1/9 | 1 | 1/47 | 46 |
| 1/19 | 18 | 1/48=1/16*1/3 | 1 |
| 1/21=1/3*1/7 | 6 | 1/49=1/7*1/7 | 42 |
| 1/22=1/2*1/11 | 2 | 1/51=1/3*1/17 | 16 |
| 1/23 | 22 | 1/52=1/4*1/13 | 6 |
| 1/24=1/8*1/3 | 1 | 1/53 | 13 |
| 1/26=1/2*1/13 | 6 | 1/54=1/6*1/9 | 3 |
| 1/27 | 3 | 1/55=1/5*1/11 | 2 |
| 1/28=1/4*1/7 | 6 | 1/56=1/8*1/7 | 6 |
| 1/29 | 28 | 1/57=1/3*1/19 | 18 |
| 1/30=1/10*1/3 | 1 | 1/58=1/2*1/29 | 28 |
| 1/31 | 15 | 1/59 | 58 |
| 1/33=1/3*1/11 | 2 | 1/60=1/20*1/3 | 1 |
| 1/34=1/2*1/17 | 16 | 1/61 | 60 |
| 1/35=1/5*1/7 | 6 | 1/62=1/2*1/31 | 15 |

By observing Table 1, we find that their least positive periods have not distinct regularity. According to Euler theorem, we will present the computation formula of period of recurring decimal, relation of the period and least positive period, and the necessary and sufficient condition that the period is eaqual to least positive period.

## II. PRELIMINARIES

**Definition 1** Let $a/b$ be an irreducible proper fraction, when it can be denoted as a recurring decimal, the repetend digit of the recurring decimal(including pure recurring decimal and mixed recurring decimal) is called the period of $a/b$, and it is denoted by $T(a/b)$; the least repetend digit is called least positive period of $a/b$, and it is denoted by $\underline{T}(a/b)$.

**Lemma 1**[3] Let $a/b$ be an irreducible proper fraction, where

$$b = 2^s 5^t, s \geq 0, t \geq 0, s+t > 0,$$

then $a/b$ can be denoted as a limited decimal.

**Lemma 2**[3] Suppose that $a/b$ is an irreducible proper fraction, and $(b,10)=1$, then

(i) $a/b$ can be denoted as a pure recurring decimal.

(ii) If $n_0$ is least positive integer such that

$$10^n \equiv 1(\mod b),$$

then $\underline{T}(a/b) = n_0$.

**Lemma 3**[3] Suppose that $a/b$ is an irreducible proper fraction,

where

$$b = 2^s 5^t b_1, (b_1, 10) = 1, s \geq 0, t \geq 0, s + t > 0,$$

$h = \max\{s, t\}$, then

(i) $a/b$ can be denoted as a mixed recurring decimal.

(ii) The digit of noncyclic part is $h$ in the decimal part of mixed recurring decimal.

(iii) If $n_0$ is least positive integer such that

$$10^n \equiv 1 (\mod b_1),$$

then $\underline{T}(a/b) = n_0$.

## III. THE MAIN RESULTS

**Theorem 1** Assume that $a/b$ is an irreducible proper fraction, $b = \prod_{i=1}^{n} p_i^{k_i}$, $k_i \geq 1$, $(p_i, 10) = 1$, and $p_i$ is prime, then we have that

(i) $T(a/b) = \varphi(b) = \prod_{i=1}^{n} p_i^{k_i-1}(p_i - 1)$ and

$\underline{T}(a/b) | T(a/b)$ ;

(ii) $\underline{T}(a/b) = T(a/b)$ if and only if

$10^i \not\equiv 10^j (\mod b)$, for any $i, j (0 \leq i < j \leq \varphi(b) - 1)$.

**Proof.** (i) Since $k_i \geq 1$, $(p_i, 10) = 1$, and $p_i$ is prime, thus

$$(p_i^{k_i}, 10) = 1, \text{i.e. } (b, 10) = 1.$$

From Euler theorem[4], we have that

$$10^{\varphi(b)} \equiv 1 (\mod b), \text{i.e. } b | (10^{\varphi(b)} - 1).$$

Let $10^{\varphi(b)} - 1 = bk$ ($k$ denotes integer), we know from $a < b$ that $ak < bk$, hence we can set

$$ak = \overline{a_1 a_2 L\ a_{\varphi(b)}} (0 \leq a_i \leq 9),$$

where $\overline{a_1 a_2 L\ a_{\varphi(b)}}$ denotes the positive integer composed of $a_1, a_2, L, a_{\varphi(b)}$.

Note that

$$(10^{\varphi(b)} - 1)0.\overset{\bullet}{a_1} a_2 L\ \overset{\bullet}{a_{\varphi(b)}}$$

$$= a_1 a_2 L\ a_{\varphi(b)}.\overset{\bullet}{a_1} a_2 L\ \overset{\bullet}{a_{\varphi(b)}} - 0.\overset{\bullet}{a_1} a_2 L\ \overset{\bullet}{a_{\varphi(b)}}$$

$$= \overline{a_1 a_2 L\ a_{\varphi(b)}},$$

then we can obtain

$$\frac{a}{b} = \frac{ak}{bk} = \frac{\overline{a_1 a_2 L\ a_{\varphi(b)}}}{10^{\varphi(b)} - 1} = 0.\overset{\bullet}{a_1} a_2 L\ \overset{\bullet}{a_{\varphi(b)}}.$$

Therefore $T(a/b) = \varphi(b)$.

In addition, since

$$\varphi(b) = \varphi(\prod_{i=1}^{n} p_i^{k_i}) = \prod_{i=1}^{n} p_i^{k_i-1}(p_i - 1) \text{[4]},$$

so we have that

$$T(a/b) = \varphi(b) = \prod_{i=1}^{n} p_i^{k_i-1}(p_i - 1).$$

Let $n_0$ be minimum positive integer such that

$$10^n \equiv 1 (\mod b),$$

then from Euler theorem, we further know that $n_0 | \varphi(b)$.

By combining known conditions and Lemma 2(ii), we derived

$$\underline{T}(a/b) = n_0.$$

Consequently, we conclude that $\underline{T}(a/b) | T(a/b)$.

(ii) If $\underline{T}(a/b) = T(a/b)$, by Theorem 1 (i), we have $\underline{T}(a/b) = \varphi(b)$.

Supposing that there are $i, j (0 \leq i < j \leq \varphi(b) - 1)$ such that

$$10^i \equiv 10^j (\mod b),$$

we then have

$$b | 10^i (10^{j-i} - 1).$$

From $(10, b) = 1$, we obtain $b | (10^{j-i} - 1)$, that is, $10^{j-i} \equiv 1 (\mod b)$.

Note that Lemma 2(ii) and $1 \leq j - i < \varphi(b)$, obviously, this is in conflict with $\underline{T}(a/b) = \varphi(b)$. Therefore, we have $10^i \not\equiv 10^j (\mod b)$, for any $i, j (0 \leq i < j \leq \varphi(b) - 1)$.

Conversely, if $10^i \not\equiv 10^j (\mod b)$, for any $i, j (0 \leq i < j \leq \varphi(b) - 1)$. Specially, by setting $i = 0$, we have for any $j (1 \leq j \leq \varphi(b) - 1)$ that

$$10^j \not\equiv 1 (\mod b),$$

Moreover, from Euler theorem, we know that

$$10^{\varphi(b)} \equiv 1 (\mod b),$$

consequently, we have $\underline{T}(a/b) = \varphi(b)$.

Combining Theorm 1(i), we can obtain

$$\underline{T}(a/b) = T(a/b).$$

The proof is completed.

**Theorem 2** Let $a/b$ be an irreducible proper fraction, where

$$b = 2^s 5^t b_1, \ s \geq 0, t \geq 0, \quad s+t > 0, \ b_1 = \prod_{i=1}^n p_i^{k_i},$$

$$k_i \geq 1, (p_i, 10) = 1,$$

and $p_i$ is prime, then we have that

(i) $T(a/b) = \varphi(b_1) = \prod_{i=1}^n p_i^{k_i-1}(p_i - 1)$ and

$$\underline{T}(a/b) \,|\, T(a/b);$$

(ii) $\underline{T}(a/b) = T(a/b)$ if and only if

$$10^i \not\equiv 10^j (\mod b_1), \text{ for any } i, j (0 \leq i < j \leq \varphi(b_1) - 1).$$

**Proof.** From known conditions and Lemma 3, we know that $\underline{T}(a/b)$ is only related to $b_1$, therefore, Theorem 2 can be proved by the proof method of Theorem 1. Here, we no longer prove the proposition.

**Theorem 3** If $(b_1, b_2) = 1$, then

$$T(1/b_1)T(1/b_2) = T(1/b_1 b_2).$$

**Proof.** From $(b_1, b_2) = 1$, we obtain

$$\varphi(b_1)\varphi(b_2) = \varphi(b_1 b_2) \text{ [4]}.$$

And by Thorem 1(i), we have that $T(1/b_1) = \varphi(b_1)$, $T(1/b_2) = \varphi(b_2)$ and $T(1/b_1 b_2) = \varphi(b_1 b_2)$.

Consequently, we have

$$T(1/b_1)T(1/b_2) = T(1/b_1 b_2).$$

The desired result follows.

According to the above conclusion, we can also obtain the following result.

**Corollary** If $(b_i, b_j) = 1$, then

$$\prod_{i=1}^n T(1/b_i) = T(1/\prod_{i=1}^n b_i).$$

Meanwhile, combine with the relevant data in table 1 and the above conclusions, we make the following guess

**Guess** If $(b_i, b_j) = 1$, then

$$\underline{T}(1/\prod_{i=1}^n b_i) \,|\, \prod_{i=1}^n \underline{T}(1/b_i).$$

IV. CONCLUSIONS AND PROSPECT

Through the above research, we have proposed the computation formulas of period of pure recurring decimal and mixed recurring decimal respectively, relation of the period and least positive period, and the necessary and sufficient condition that the period is eaqual to least positive period.But we think that the above conclusions are not perfect, in future studies, we will further weaken the sufficient and necessary conditions, and discuss the calculation formula of least positive period of recurring decimal and other beautiful properties.

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